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THE EFFECTIVE ELASTICITY TENSORS FOR DISPERSED COMPOSITES†

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Analytic formulae based on periodic fundamental solutions are obtained for the correction tensor and the effective elasticity tensor of dispersed statistically homogeneous elastic composites.

THE MOST vigorous results in the mechanics of composites can be obtained by the method of two-scale asymptotic expansions [1–5]. In this method, the effective elasticity tensor is usually represented in the form

$$\mathbf{C}_0 = \sum_{p=1}^{N} f_p \mathbf{C}_p + \mathbf{K}, \quad \sum_{p=1}^{N} f_p = 1 \tag{0.1}$$

where f_p is the fraction by volume of a component of the N-component composite, $\mathbf{C}_p$ is the corresponding elasticity tensor, and $\mathbf{K}$ is the correction tensor, or corrector. The sum of the first N terms on the right-hand side of (0.1) gives the effective tensor obtained by Voigt homogenization. We shall consider below the case of a two-component composite for which $f_1 = f$, $f_2 = (1-f)$, where f is the volume fraction of the dispersed component.

To determine the correction tensor in the method of two-scale asymptotic expansion it is necessary to solve the so-called cell problem, i.e. construct a periodic solution of the equations of the theory of elasticity in a cell.

As well as difference methods of the finite-element and finite-difference types, the following (numerical) methods exist for solving the cell problem. The method of Eshelby transformation strain, taking account of the variability of the field of transformation strain within the inclusion, has been used in [6–9]. One of the advantages of this approach is the possibility of analysing composites with anisotropic components. However, from the numerical point of view they are unsatisfactory because of the need to solve a system of three-dimensional integral equations of the first kind with tensor density fields of the transformation strain. Using the periodic fundamental solution for an isotropic medium [10] numerical values of the effective characteristics of dispersionally reinforced composites with isotropic components were obtained in [11, 12] by multiple expansions. A similar fundamental solution was used in [13] combined with the Galerkin method to solve a system of boundary integral equations on the surface separating the components. Boundary conditions for an initially isotropic porous medium were proposed in [14] for the surfaces of the pores and periodic boundary conditions on the external surface of the cell are satisfied by solving a system of integral equations of the first kind with a kernel that is a fundamental Kelvin solution.

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1. FUNDAMENTAL OPERATORS AND SYMBOLS

Consider an initially anisotropic homogeneous elastic medium whose equilibrium equations in R^3 have the form

$$\mathbf{A}(\partial_x)\mathbf{u} \equiv -\operatorname{div}\mathbf{C}\cdot\cdot(\nabla\mathbf{u}) \tag{1.1}$$

where $\mathbf{u}$ is the displacement vector and $\mathbf{C}$ is a fourth-order strictly elliptic elasticity tensor. It is assumed that the medium investigated is hyperelastic, which ensures the symmetry of $\mathbf{C}$ with respect to outer pairs of indices: $C^{ijmn} = C^{mnij}$.

The Fourier transformation

$$g^{\vee}(\xi) = \int_{R^3} g(\mathbf{x})\exp(-2\pi i\mathbf{x}\cdot\xi)\,dx$$

applied to the operator $\mathbf{A}$ gives the corresponding symbol

$$\mathbf{A}^{\vee}(\xi) = 4\pi^2\xi\cdot\mathbf{C}\cdot\xi \tag{1.2}$$

The symbol of the stress operator on the boundary manifold with normal field υ is defined similarly

$$\mathbf{T}^{\vee}(\upsilon,\xi) = 2\pi i\boldsymbol{\upsilon}\cdot\mathbf{C}\cdot\xi \tag{1.3}$$

Using the symbol $\mathbf{A}^{\vee}$ and the definition of the fundamental solution $\mathbf{E}$ of Eq. (1.1), the symbol $\mathbf{E}^{\vee}$ can be represented in the form

$$\mathbf{E}^{\vee}(\xi) = \mathbf{A}_0^{\vee}(\xi)/\det\mathbf{A}^{\vee}(\xi) \tag{1.4}$$

where $\mathbf{A}_0^{\vee}$ is the matrix of the cofactors of the symbol $\mathbf{A}^{\vee}$. Formula (1.4) shows that the symbol $\mathbf{E}^{\vee}$ is strictly elliptic and positively homogeneous in ξ of degree -2. For the general anisotropic case, only numerical methods of reconstructing $\mathbf{E}$ from its symbol are known [15], but a periodic fundamental solution can be constructed directly from the symbol $\mathbf{E}^{\vee}$.

To construct the periodic fundamental solution $\mathbf{E}_p$ we consider a medium with force singularities, periodically distributed at the nodes of some spatial grid Λ. Suppose $\mathbf{a}_i$ ($i = 1, 2, 3$) are linearly-independent vectors of the principal periods of Λ, so that we can represent any node $\mathbf{m}\in\Lambda$ in the form $\mathbf{m} = \Sigma m_i\mathbf{a}_i$, where the $m_i \in Z$ are integer-valued coordinates of the node $\mathbf{m}$ in the periodic basis $(\mathbf{a}_i)$.

We introduce into consideration the conjugate basis $(\mathbf{a}_i{}^*)$ such that $\mathbf{a}_i{}^*\cdot\mathbf{m} = m_i$. It is clear that for mutually orthogonal fundamental basis vectors the conjugate basis vectors are directed along the corresponding vectors of the fundamental basis. The grid of the conjugate basis will be denoted by Λ^*. Using this notation the periodic delta-function (δ_p) distributed along the nodes of the grid Λ can be expanded in series

$$\delta_p(\mathbf{x}) = V_Q^{-1}\sum_{\mathbf{m}^*\in\Lambda^*}\exp(-2\pi i\mathbf{m}^*\cdot\mathbf{x}) \tag{1.5}$$

where V_Q is the volume of the fundamental domain (periodicity cell) formed by the vectors of the fundamental periodic basis $V_Q = l_1 l_2 l_3 |\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3|$. Formula (1.5) defines the function δ_p uniquely.

Substituting the periodic fundamental solution $\mathbf{E}_p$ into Eq. (1.1) we obtain

$$\mathbf{A}(\partial_x)\mathbf{E}_p = \delta_p\mathbf{I} \tag{1.6}$$

where $\mathbf{I}$ is the unit diagonal matrix. Looking for $\mathbf{E}_p$ also in the form of a trigonometric series, from (1.5), (1.6) we obtain a formula from which $\mathbf{E}_p$ can be computed apart from a constant tensor

$$\mathbf{E}'_p(\mathbf{x}) = V_Q^{-1} \sum_{\mathbf{m}^* \in \Lambda_0^*} \check{\mathbf{E}}(\mathbf{m}^*) \exp(-2\pi i \mathbf{m}^* \cdot \mathbf{x}) \tag{1.7}$$

where $\Lambda_0{}^*$ is the grid of the conjugate basis without the zero node. We have the following result [16].

Lemma 1. Series (1.7) converges in the L_1 topology to a periodic fundamental solution of class $\overline{L}^1(Q,\ R^3 \otimes R^3)$, where $\overline{L}^1$ is the space of integrable functions with zero mean in Q. Furthermore

$$\mathbf{E}'_p(\mathbf{x}) = \mathbf{E}(\mathbf{x}) + \mathbf{G}(\mathbf{x}),\quad \mathbf{G} \in C^\infty(Q, R^3 \otimes R^3) \tag{1.8}$$

Note that series (1.7) does not converge absolutely for any $\mathbf{x} \in R^3$ [17].

We denote by Ω the disconnected domain occupied by periodically dispersed inclusions. Suppose χ_Ω is the characteristic function of this domain. The elastic properties of the two-component heterogeneous medium can be represented in the form

$$\mathbf{C}_1 \chi_\Omega(\mathbf{x}) + \mathbf{C}_2 \chi_{C\Omega}(\mathbf{x}) = \mathbf{C}_1 + \mathbf{C}\chi_{C\Omega}(\mathbf{x}),\quad \mathbf{x} \in R^3, C\Omega = R^3 \setminus \Omega,\quad \mathbf{C} = \mathbf{C}_2 - \mathbf{C}_1 \tag{1.9}$$

where the subscript 1 refers to the dispersed inclusions, and 2 to the matrix. The right-hand side of Eq. (1.9) shows that for the approximate determination of the effective tensor $\mathbf{C}_0$ it is sufficient to determine the effective tensor of the porous medium with elasticity $\mathbf{C}$ and pores occupying the domain Ω

$$\mathbf{C}_0 = \mathbf{C}_1 + (1 - f)\mathbf{C} + \mathbf{K} \tag{1.10}$$

where $\mathbf{K}$ is the correction tensor of the porous medium. Below we shall assume that the tensor $\mathbf{C} = \mathbf{C}_2 - \mathbf{C}_1$ is strictly elliptic. To determine the tensor $\mathbf{K}$ we will use the method of two-scale asymptotic expansions.

2. ASYMPTOTIC EXPANSIONS

We will represent the displacement field in a periodic porous medium in the form of an asymptotic expansion

$$\mathbf{u}(\mathbf{x}, \mathbf{Y}) = \sum_{n=0}^{\infty} \lambda^n \mathbf{u}_n(\mathbf{x},\ \mathbf{Y}),\quad \mathbf{Y} = \mathbf{x}/\lambda \tag{2.1}$$

where the $\mathbf{Y}$ are "fast" variables characterizing the oscillations of the field $\mathbf{u}$.

Changing to variables $\mathbf{x}, \mathbf{Y}$ in (1.1) we obtain [5]

$$\mathbf{A}(\partial_x, \partial_Y) \equiv \lambda^{-2}\mathbf{A}_1(\partial_Y) + \lambda^{-1}\mathbf{A}_2(\partial_x, \partial_Y) + \lambda^0 \mathbf{A}_3(\partial_x) \tag{2.2}$$

$$\mathbf{A}_1(\partial_Y) \equiv -\nabla_Y \cdot \mathbf{C}(\mathbf{Y}) \cdot \nabla_Y,\quad \mathbf{A}_2(\partial_x, \partial_Y) \equiv -\nabla_x \cdot \mathbf{C}(\mathbf{Y}) \cdot \nabla_Y - \nabla_Y \cdot \mathbf{C}(\mathbf{Y}) \cdot \nabla_x$$

$$\mathbf{A}_3(\partial_x) \equiv -\nabla_x \cdot \mathbf{C}(\mathbf{Y}) \cdot \nabla_x,\quad \mathbf{C}(\mathbf{Y}) = \begin{cases} \mathbf{C}, & \mathbf{Y} \in Q\setminus\Omega \\ 0, & \mathbf{Y} \in \Omega \end{cases}$$

In expressions (2.2) and below the variables $\mathbf{x}$ and $\mathbf{Y}$ are taken to be independent.

Substituting the asymptotic series (2.1) into (2.2) we obtain

$$\mathbf{A}_1(\partial_Y)\mathbf{u}_0 = 0, \quad \mathbf{A}_1(\partial_Y)\mathbf{u}_1 = -\mathbf{A}_2(\partial_x, \partial_Y)\mathbf{u}_0 \tag{2.3}$$

$$\mathbf{A}_1(\partial_Y)\mathbf{u}_2 = -\mathbf{A}_2(\partial_x, \partial_Y)\mathbf{u}_1 - \mathbf{A}_3(\partial_x)\mathbf{u}_0, \ldots$$

The first three equations of (2.3), corresponding to λ^{-2}, λ^{-1}, λ^{0}, are of greatest interest for finding the effective characteristics of the porous medium. The remaining equations are only necessary when analysing the behaviour of the field microstructure in the vicinity of boundaries or in domains with large gradients (with respect to $\mathbf{x}$).

We denote by W the subspace of $H^1(Q\backslash\Omega, R^3\otimes R^3)$ such that the condition $\Phi\in W$ is equivalent to the periodicity of Φ and $\mathbf{T}(\upsilon_Y, \partial_Y)\Phi|_{\partial\Omega}=0$, where $\mathbf{T}$ is the stress operator on $\partial\Omega$ with the outward unit normal vector υ directed away from $Q\backslash\Omega$.

Lemma 2. In order for the equation $\mathbf{A}_1(\partial_Y)\Phi=\mathbf{F}$, $\Phi\in W$, $\mathbf{F}\in H^{-1}$ to be solvable $(\mathrm{mod}\, R^3)$, it is necessary and sufficient that the mean "value" of $\mathbf{F}$ in $Q\backslash\Omega$ be zero.

Proof. The second Korn inequality, whose proof by virtue of (1.8) and the methods of integral equations reduces to the non-periodic case, ensures the coerciveness of the bilinear form

$$a_1(\Phi, \Psi) = \int_{Q\backslash\Omega}\nabla_Y\Phi \cdot\cdot\, \mathbf{C} \cdot\cdot\, \nabla_Y\Psi dY$$

in W/R^3. The need to consider the factor-space W/R^3 is due to the existence of non-trivial periodic solutions of the equation $\mathrm{sym}(\Delta\Phi)=0$. Here, unlike in the non-periodic case, such solutions are affine displacements. One should also note that for $\Phi\in W$

$$\int_{Q\backslash\Omega}\mathbf{A}_1(\partial_Y)\Phi dY = 0$$

Corollary 1. The unique periodic solution of the first equation of (2.3) in $Q\backslash\Omega$ is a solution of the form $\mathbf{u}_0=(\mathbf{x}, \mathbf{Y})=\mathbf{u}_0(\mathbf{x})$.

2. The general solution of the second equation of (2.3) in $Q\backslash\Omega$ has the form

$$\mathbf{u}_1(\mathbf{x}, \mathbf{Y}) = \mathbf{H}(\mathbf{Y}) \cdot\cdot\, \epsilon_0(\mathbf{x}) + \tilde{\mathbf{u}}_1(\mathbf{x}), \quad \epsilon_0 = \mathrm{sym}(\nabla\mathbf{u}_0) \tag{2.4}$$

where $\mathbf{H}$ is a third-order tensor field, being the solution of the equation $\mathbf{A}_1\mathbf{H}=0$ and $Q\backslash\Omega$, and $\mathbf{TH}|_{\partial\Omega}=-\upsilon\cdot\mathbf{C}$.

3. For the third equation of (2.3) to be solvable it is necessary and sufficient that

$$\int_{Q\backslash\Omega}\mathbf{A}_3(\partial_x)\mathbf{u}_0 dY + \int_{Q\backslash\Omega}\mathbf{A}_2(\partial_x, \partial_Y)\mathbf{u}_1 dY = 0 \tag{2.5}$$

It follows from (2.4) that

$$\int_{Q\backslash\Omega}\mathbf{A}_2(\partial_x, \partial_Y)\mathbf{u}_1 dY = -\mathrm{div}_x\int_{Q\backslash\Omega}\mathbf{C} \cdot\cdot\, \nabla_Y\mathbf{H}(\mathbf{Y})dY \cdot\cdot\, \epsilon_0 \tag{2.6}$$

To obtain this expression we took into account that the term $\tilde{\mathbf{u}}_1$ does not occur in the third equation of (2.3). Bearing in mind (2.6), we obtain from (2.5)

$$(1-f)\,\mathrm{div}_x\mathbf{C} \cdot\cdot\, \epsilon_0(\mathbf{x}) + \mathrm{div}_x\mathbf{K} \cdot\cdot\, \epsilon_0(\mathbf{x}) = 0 \tag{2.7}$$

$$\mathbf{K} = -V_Q^{-1}\int_{\partial\Omega}\mathbf{C} \cdot\cdot\, \upsilon_Y \otimes \mathbf{H}(\mathbf{Y})dY$$

where f is the volume porosity coefficient. Here and below $\partial\Omega$ is taken to be an oriented manifold with orientation induced from the domain $Q\backslash\Omega$.

Equation (2.7) is the required equilibrium equation for a homogenized medium. From (2.7) it follows that the effective elasticity tensor of a porous medium has the form

$$\mathbf{C}_0' = (1-f)\mathbf{C} + \mathbf{K} \tag{2.8}$$

The first term on the right-hand side of (2.8) corresponds to Voigt homogenization.

3. DETERMINATION OF THE CORRECTOR

The Somigliana identity for $Q\backslash\Omega$ gives (initially considering in this section a porous medium with elasticity tensor $\mathbf{C}$)

$$\begin{aligned}&(1/2\mathbf{I}+\mathbf{S})\mathbf{H}(\mathbf{Y}') = \int_{\partial\Omega}\mathbf{E}_p'(\mathbf{Y}'-\mathbf{Y}'')\otimes\upsilon_{Y''}\cdot\cdot\,\mathbf{C}dY''+\mathbf{H}_a,\ \mathbf{Y}'\in\partial\Omega\\ &\mathbf{H}_a = V_Q^{-1}\int_{Q\backslash\Omega}\mathbf{H}(\mathbf{Y})dY\end{aligned} \tag{3.1}$$

where $\mathbf{S}$ is a singular matrix operator obtained by restricting the double-layer potential on the carrying surface $\partial\Omega$. It is necessary to introduce $\mathbf{H}_a$ on the right-hand side of (3.1) because the tensor $\mathbf{E}_p'$ is defined by formula (1.7), apart from a constant vector.

Lemma 3. If the domain Ω is centrally symmetric about the origin of coordinates then

$$\int_{\partial\Omega}\int_{\partial\Omega}\mathbf{E}_p'(\mathbf{Y}'-\mathbf{Y}'')\otimes\upsilon_{Y''}\cdot\cdot\,\mathbf{C}dY'dY''=0 \tag{3.2}$$

Proof. By (1.7) condition (3.2) is equivalent to

$$2\pi i V_Q^{-1}\sum_{\mathbf{m}^*\in\Lambda_0^*}\mathbf{E}^{\vee}(\mathbf{m}^*)\otimes\mathbf{m}^*\cdot\cdot\,\mathbf{C}\chi_{\partial\Omega}^{\vee}(\mathbf{m}^*)\,\chi_{\Omega}^{\vee}(\mathbf{m}^*)=0 \tag{3.3}$$

where $\chi_{\partial\Omega}$, χ_Ω are the characteristic functions of $\partial\Omega$ and Ω, respectively. To prove (3.3) it is sufficient to note that the characteristic functions under consideration, just like their Fourier-transforms, are even, so that the symbol $\mathbf{E}^{\vee}$ is also even.

Definition. The spectrum of the operator $\mathbf{S}$ is the set of those λ for which the operator $\lambda\mathbf{I}=\mathbf{S}$ is non-invertible in the class of continuous operators acting in the appropriate functional space.

This definition is identical with that used in spectral theory and does not differ significantly from the definition of a spectrum in the theory of integral equations. Analysis of the periodic solutions of the second boundary-value problem with surface stresses specified on $\partial\Omega$ shows that the points $|\lambda|=\frac{1}{2}$ lie outside the spectral circle of the operator $\mathbf{S}$ acting in the Sobolev spaces $\overline{H}^s(\partial\Omega, R^3)$, $s>0$ of functions with zero mean on $\partial\Omega$. However, in the spaces H^s the spectral circle already contains the point $\lambda=\frac{1}{2}$, with a corresponding spectral space consisting of "rigid" displacements of the contour.

From the Somigliana identity Lemma 3 and subsequent remarks we have the following lemma.

Lemma 4. Under the conditions of Lemma 3 the Neumann series

$$(\tfrac{1}{2}\mathbf{I}+\mathbf{S})^{-1} = 2\sum_{n=0}^{\infty}(-2\mathbf{S})^n \tag{3.4}$$

converges absolutely in the operator topology $(\overline{H}^s, \overline{H}^s)$, $s\geqslant 0$.

In the statement of the lemma, $(-2\mathbf{S})^n$ is the matrix integral operator that is the composition of n singular integral operators $(-2\mathbf{S})$. The rate of convergence of this series can be estimated in terms of the majorant $p=\|2\mathbf{S}\|_s$. If $p<1$ series (3.4) converges faster than a geometrical series with common ratio p.

Substituting expressions (3.1) and (3.4) into the expression for the corrector (2.7) and transforming the surface integrals over $\partial\Omega$ into volume integrals, we obtain

$$\mathbf{K} = \sum_{n=0}^{\infty}(-2)^{n+1}(2\pi)^{2n+2}V_Q^{-n-2}\times\Big(\sum_{\mu^*\in\Pi_n}\chi_\Omega^{\vee}(\mu_0^*)\chi_\Omega^{\vee}(\mu_1^*-\mu_0^*)\ldots$$

$$\ldots \chi^{\vee}_{\Omega}(\mu_n^* - \mu_{n-1}^*)\chi^{\vee}_{\Omega}(-\mu_n^*) \times \mathbf{C} \cdot\cdot \mu_0^* \otimes \mathbf{E}^{\vee}(\mu_0^*) \otimes \mu_0^* \cdot\cdot \mathbf{C} \cdot\cdot \ldots$$
$$\ldots \cdot\cdot \mathbf{C} \cdot\cdot \mu_n^* \otimes \mathbf{E}^{\vee}(\mu_n^*) \otimes \mu_n^* \cdot\cdot \mathbf{C}), \tag{3.5}$$
$$\mu_p^* \in \Lambda_0^*,\ p = 0, \ldots, n,\ \Pi_n = \prod_{p=0}^{n} \Lambda_0^*$$

Expression (3.5) is the required formula for the correction tensor. Bearing in mind the alternating sign of the terms of the series with respect to n and restricting ourselves to the first term corresponding to $n = 0$, we obtain a lower approximation for the corrector

$$\mathbf{K}_l = -8\pi^2 V_Q^{-2} \sum_{\mathbf{m}^* \in \Lambda_0^*} |\chi^{\vee}_{\Omega}|^2 \mathbf{C} \cdot\cdot \mathbf{m}^* \otimes \mathbf{E}^{\vee}(\mathbf{m}^*) \otimes \mathbf{m}^* \cdot\cdot \mathbf{C} \tag{3.6}$$

Similarly, if we restrict ourselves to the first two terms of the series in n, one can easily obtain an upper estimate for the correction tensor from (3.5)

$$\mathbf{K}_u = -8\pi^2 V_Q^{-2} \sum_{\mathbf{m}^* \in \Lambda_0^*} |\chi^{\vee}_{\Omega}|^2 \mathbf{C} \cdot\cdot \mathbf{m}^* \otimes \mathbf{E}^{\vee}(\mathbf{m}^*) \otimes \mathbf{m}^* \cdot\cdot \mathbf{C} +$$
$$+ 4(2\pi)^4 V_Q^{-3} \sum_{\mathbf{m}_0^* \in \Lambda_0^*} \sum_{\mathbf{m}_1^* \in \Lambda_0^*} \chi^{\vee}_{\Omega}(\mathbf{m}_0^*)\chi^{\vee}_{\Omega}(-\mathbf{m}_1^*)\chi^{\vee}_{\Omega}(\mathbf{m}_1^* - \mathbf{m}_0^*) \times$$
$$\times \mathbf{C} \cdot\cdot \mathbf{m}_0^* \otimes \mathbf{E}^{\vee}(\mathbf{m}_0^*) \otimes \mathbf{m}_0^* \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{m}_1^* \otimes \mathbf{E}^{\vee}(\mathbf{m}_1^*) \otimes \mathbf{m}_1^* \cdot\cdot \mathbf{C} \leqslant$$
$$\leqslant -8\pi^2 V_Q^{-2} \sum_{\mathbf{m}^* \in \Lambda_0^*} |\chi^{\vee}_{\Omega}|^2 \mathbf{C} \cdot\cdot \mathbf{m}^* \otimes \mathbf{E}(\mathbf{m}^*) \otimes \mathbf{m}^* \cdot\cdot \mathbf{C} +$$
$$+ 4(2\pi)^4 V_Q^{-2} f \sum_{\mathbf{m}^* \in \Lambda_0^*} |\chi^{\vee}_{\Omega}(\mathbf{m}^*)|^2 \mathbf{C} \cdot\cdot \mathbf{m}^* \otimes \mathbf{E}^{\vee}(\mathbf{m}^*) \otimes \mathbf{m}^* \cdot\cdot \mathbf{C} =$$
$$= -8\pi^2 V_Q^{-2}(1-2f) \sum_{\mathbf{m}^* \in \Lambda_0^*} |\chi^{\vee}_{\Omega}|^2 \mathbf{C} \cdot\cdot \mathbf{m}^* \otimes \mathbf{E}^{\vee}(\mathbf{m}^*) \otimes \mathbf{m}^* \cdot\cdot \mathbf{C} \tag{3.7}$$

In obtaining (3.7) we used Young's inequality for convolutions.

Theorem. Series (4.5)–(4.7) converge absolutely.

The proof follows from the asymptotic estimate

$$|\chi^{\vee}_{\Omega}(|\xi|)| = o(|\xi|^{-3/2}),\ |\xi| \to \infty, \tag{3.8}$$

which is satisfied since $\chi_\Omega \in L^2(R^3)$. It remains to note that the symbol $\mathbf{C} \cdot\cdot \mathbf{m}^* \otimes \mathbf{E}^{\vee}(\mathbf{m}^*) \otimes \mathbf{m}^* \cdot\cdot \mathbf{C}$ is positively homogeneous of degree zero with respect to $\mathbf{m}^*$.

Note that the transition from the effective tensor of the porous medium $\mathbf{C}_0'$ to the corresponding effective tensor of the composite is given according to (1.10) and (2.8) by the formula $\mathbf{C}_0 = \mathbf{C}_1 + \mathbf{C}_0'$.

4. EXAMPLE

Consider a dispersed composite with an isotropic matrix characterized by dimensionless elastic parameters (Young's modulus and Poisson's ratio) $E_2 = 1$ and $\nu_2 = 0.3$ and spherical inclusions with $E_1 = 0.01$ and $\nu_1 = 0.45$, distributed at the nodes of a simple cubic lattice. The limiting packing coefficient f for the lattice (the volume fraction of the dispersed inclusions) is $\pi/6 \cong 0.52$. A composite with this kind of inclusion serves as a model of certain shock-proof plates.

Figure 1 shows numerical results giving the dependence of the effective moduli C_0^{1122} (curve 1) and C_0^{1212} (curve 2) on the packing coefficient (f). The calculation was performed using (3.6). For comparison, similar graphs of the Lamé constants λ_0 and μ_0 were constructed for the effective isotropic medium obtained by the Voigt homogenization method.

Because the simple cubic structure necessarily leads to cubic anisotropy in the ratios of the elastic

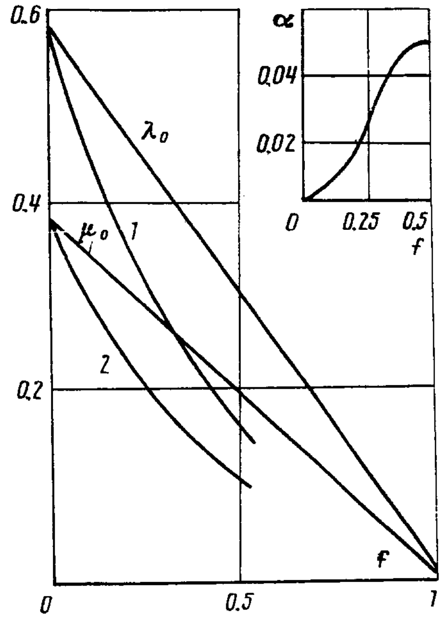

Fig. 1.

properties, the figure shows the dependence of the parameter α

$$\alpha = C_0^{1212} - (C_0^{1111} - C_0^{1122})/2 \tag{4.1}$$

which describes the degree of anisotropy of the effective elastic medium, on the packing coefficient. It follows directly from (4.1) that for the original isotropic medium without inclusions $\alpha = 0$.

The processor time on an IBM PC/AT-286 (12 MHz) computer required for 2, 4, 5, and 6 nodes was 100, 600, 10^3 and 1.8×10^3 seconds, respectively.

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